



## Nonlinear Analysis in Economic & Financial

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### ABSTRACT

Traditional economic and financial models often use linear frameworks to analyze the relationships between prices and fundamentals. For instance, the cost of carry model suggests that stock futures prices should align with spot prices (Taylor et al., 2000), while log-linear present value models propose a linear relationship between log dividends and stock prices (Campbell and Shiller, 1988). Similarly, housing market models predict stability in the house price-to-income ratio if residential property values are consistently proportional to future real disposable income (Black et al., 2006). However, empirical research using linear unit root tests frequently shows that deviations from fundamental values do not revert to equilibrium as theory suggests. This empirical discrepancy indicates that linear models and tests might be inadequate in capturing real-world dynamics. Recent studies propose that nonlinear dynamics might be a key factor in this misalignment, suggesting that linear unit root tests could yield misleading results when nonlinear mechanisms are at play in macroeconomic and asset markets.

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**Introduction:** Traditional models in economics and finance establish a linear framework for analyzing prices, fundamentals as well as the relationship between them. Regarding stock markets, the cost of carry model predicts that stock futures prices should comove with spot prices (Taylor et al., 2000). Further, loglinear present value models imply a linear relationship between log dividends and prices (Campbell and Shiller, 1988). Similarly in the housing market, as long as that the real value of residential property is a constant proportion of the expected value of future real disposable income, the house price to income ratio should be stable (Black et al., 2006). However, in most cases the results of empirical research based on linear unit root tests are discouraging. Deviations from fundamental values appear to wonder with no apparent tendency to revert to a single equilibrium point. This is in sharp contrast to the

theory. During the last two decades numerous theoretical and empirical contributions have attempted to provide possible explanations for this empirical regularity. An important finding of these studies is that the failure to find economically meaningful results may be attributed to the assumption of linearity. Linear unit root tests may result in misleading inference in the presence of nonlinear dynamics regarding the mechanism characterizing the macroeconomy and asset markets.

Second, this particular type of nonlinear models can display multiple equilibria. This property might turn out to be crucial for capturing the salient aspects of series such as real interest rates and inflation. Models of monetary policy rules suggest that once you take into account the zero bound on nominal interest rates, real interest rates might follow a number of equilibria (see, e.g., Benhabib et al., 2001). Moreover, in their seminal paper Sargent and Wallace (1973) show that when authorities print money so as to finance a real budget deficit inflation is a nonlinear process with two equilibria, one stable and the other unstable.<sup>1</sup> Moreover, a stylized fact regarding inflation is the high persistence of the series. In numerous empirical and theoretical contributions inflation is assumed to be a linear integrated of order one,  $I(1)$ , process. This assumption has severe economic and statistical implications. The nominal exchange rate, via PPP, should be an  $I(2)$  process, and nominal asset returns would exhibit unit root behavior, which is in sharp contrast with the empirical findings.

Given the widespread application of ESTAR models to time series which are highly persistent it is useful to examine the properties of a unit root test which derives from this family, and compare it with more general unit root tests. Kapetanios et al. (2003) (KSS hereafter) propose a testing procedure so as to formally distinguish between nonlinear single equilibria globally stationary processes and unit root processes. On the basis of Monte Carlo simulation experiments the authors conclude that their procedure improves substantially upon existing tests. In particular, their nonlinear unit root test appears to have better power properties than the standard Augmented Dickey-Fuller test, ADF hereafter, and the statistic proposed by Enders and Granger (1998), EG hereafter. In Section 3 of this paper we attempt to extend the analysis of KSS in three directions. First, we employ a different range of parameter values than the ones considered initially by Kapetanios et al.. The new range approximates the range of values typically reported in the empirical literature and corresponds to ESTAR processes which exhibit more closely unit root like behavior. Second, we examine the impact of uncertainty regarding the presence of deterministic components in the Data Generating Process (DGP) on the performance of the test. Clearly, researchers are often ignorant and in many cases have no priors about the true DGP so that they have to rely on hypothesis testing to determine the significance of deterministic components.<sup>2</sup> Finally, we investigate further the impact of multiple equilibria on the performance of the nonlinear test. This exercise is interesting given that the test is based on the assumption of a single equilibrium point.

Modeling Nonlinear Economic & Financial Dynamics:

A typical Exponential STAR (ESTAR) model for a univariate time series  $y_t$  is given by

$$y_t = \beta y_{t-1} + \gamma y_{t-1} [1 - \exp(-\theta y_{t-1}^2)] + \varphi_t,$$

or, equivalently

$$\Delta y_t = \phi y_{t-1} + \gamma y_{t-1} [1 - \exp(-\theta y_{t-1}^2)] + \varphi_t,$$

where  $\beta$ ,  $\gamma$  and  $\theta$  are constants,  $\phi = \beta - 1$  and  $\varphi_t \sim \text{iid}(0, \sigma^2)$  is the disturbance term.<sup>3</sup> When  $\gamma < 0$  and  $\phi + \gamma < 0$  the process is globally mean reverting although close to the equilibrium it may exhibit unit root or even explosive behavior. As aforementioned, the fact that STAR models allow the speed of mean reversion of the process towards the equilibrium to be a function of the distance from the equilibrium is particularly appealing in modeling several macroeconomic and financial variables. It has been proved that the presence of transactions costs and other market frictions in arbitrage models imply this type of nonlinear adjustment mechanism.

However, for different parameter values the ESTAR model can display multiple equilibria, complex dynamics and chaotic-like behavior that can lead to misleading conclusion when unit root tests are employed. For instance, consider the following model.

$$y_t = 1.1y_{t-1} \exp[-8.059(y_{t-1} - 0.175)^2] + \varphi_t$$

which is globally stationary. Figure 1, which depicts  $\Delta y_t$  against  $y_{t-1}$ , shows that the process has three equilibria which correspond to the cases where the curve intersects with the horizontal axis. The stable equilibria are given by 0 and 0.283. To shed more light on the properties of

the process, we simulate (2) without noise and a starting value of 0.1. The first 100 realizations of the series are presented in Figure 2. The process moves from the starting value to the high equilibrium with oscillations. This behavior could be mistakenly interpreted as either explosive or suggestive of a time trend in small samples. Figure 3 shows 100 observations of the same process with the noise switched on. It can be observed that the series exhibits high persistence which makes it difficult to distinguish from a process with an intercept and trend, a unit root or even an explosive process.<sup>4</sup> Lundbergh and Terasvirta (2002) conduct a similar simulation experiment " using a stationary Logistic STAR model with realizations that fluctuate between two local means. Their overall conclusion is that standard unit root tests when applied to these series do not reject the unit root hypothesis.

Macroeconomic and financial series may in fact exhibit this type of behavior. Byers and Peel (2000) motivated by the theoretical literature on inflation dynamics fit ESTAR models to the inflation series of high-inflation countries. They examine Brazil and Argentina in the second half of the twentieth century and Germany in interwar period.<sup>5</sup> Their results support the presence of multiple equilibria. In particular, Brazil has a stable high inflation equilibrium, while for the remaining two countries the high inflation state is characterized by inflation cycles. This finding has potential consequences for asset markets, such

as the FX and bond markets, through spillover effects. Exchange rates, interests rates and asset returns are linked with prices through e.g. the PPP hypothesis and the Fisher equation.

Monte Carlo Simulation, New Parameter Values KSS evaluate the power of their test by employing a DGP with speed of adjustment parameter  $\theta \in \{0.01, 0.05, 0.1, 1\}$ . The findings of recent empirical research (Taylor et al., 2001; Kilian and Taylor, 2003; Paya et al., 2003) suggest that even lower values of  $\theta$  are warranted, in particular, values around  $\theta = 0.001$  are also reported in empirical work.<sup>6</sup> We employ this value as well as the more extreme case of  $\theta = 0.0001$  which is closer to the linear unit root case. Regarding  $\gamma$  and  $\phi$ , we set them equal to -1 and 0, which are the values reported or imposed in most empirical research on PPP or other arbitrage conditions. The nominal significance level is set equal to 5 per cent for all the experiments implemented in this study and the number of replications equal to 10,000. Table 1 reports rejection rates of the unit root hypothesis corresponding to the KSS, EG and ADF statistics. Case 1, Case 2 and Case 3 correspond to DGPs with no constant, constant and constant and trend, respectively. The results for the KSS and EG tests presented in Panel A are based on the procedure advocated by KSS where there are priors regarding the nuisance parameters. Not surprisingly, for relative high values of  $\theta$ , 0.01, the power of the tNL statistic is always higher or similar to the power of the ADF and the EG. As the value of  $\theta$  decreases the power of all three statistics falls and when  $\theta$  reaches 0.001 the power is reduced up to five times. It is important that the reduction in power is generally more severe for the KSS than the ADF so that in some cases (and always for  $\theta = 0.0001$ ) the latter becomes superior to the former.

**Empirical Applications** In the introduction we highlighted the fact that numerous theoretical and empirical contributions suggest that factors such as agent heterogeneity, transactions costs, uncertainty regarding equilibrium values, or the sunk costs of international arbitrage can induce smooth transition nonlinearity in the deviation process of asset prices from their fundamental value in different asset markets (Dumas, 1992; Berka, 2005; Kilian and Taylor, 2001; Gallagher and Taylor, 2001; Kim and Bhattacharya, 2009). Many studies have further shown that these processes can be parsimoniously modelled by the ESTAR (Michael et al., 1997; Taylor et al., 2001; Kilian and Taylor, 2003; Sarno and Monoyios, 2002). However, smooth transition nonlinearity is not constrained to deviations of asset prices from fundamentals. Sargent and Wallace (1973) show that when authorities print money so as to finance a real budget deficit inflation becomes a nonlinear process with multiple equilibria. Evans et al. (1996) illustrate further that under adaptive learning a stable high inflation state may arise. To this end, Byers and Peel (2000) advocate the use of ESTAR models which allow high persistence of the series as well as multiple equilibria. Given all this evidence it seems interesting to test the properties of a number of series representative of those markets. Our data set consists of five real exchange rates, a house price-income ratio, a dividend-price ratio, a stock index basis, and an inflation series. The first real exchange

rate series is the annual dollar-sterling (quk,a) analyzed in Lothian and Taylor (1996). The series is extended by using data for the U.S. and U.K. consumer price indices and the dollar-sterling.

The second column of Table 3 shows the estimated  $\theta$  coefficients. The estimates vary considerably across applications. Moreover, all coefficients are statistically significant at least at the 10 per cent significance level implying that the series are nonlinear and globally stationary. The maximum  $\theta$  estimate is about 0.058 and corresponds to the annual dollar-sterling real exchange rate data. While, the minimum is about 0.004, which lies outside the range examined by KSS but included in the previous section, for the real exchange rate of Canada. These results have the following economic implications. Regarding real exchange rates, they suggest that prices and exchange rates are related with the adjustment mechanism being nonlinear. Moreover, they are in line with other studies which utilize nonlinear models and explain the documented difficulty of unit root tests typically employed in the 1980s to reject the null hypothesis (Michael et al., 1997). Turning to the dividend-price ratio, the fact that  $\theta$  is significant for the Nasdaq index complements the analysis of Gallagher and Taylor (2001) and rules out the presence of bubbles (the dot-com bubble) suggested by other studies. A similar conclusion can be drawn from the estimate of the speed of adjustment for the house price-income ratio of the U.K. housing market. The fact that the FTSE futures basis is nonlinearly mean reverting is in accordance with Sarno and Monoyios (2002) and stresses the importance of modeling stock market frictions.

**Conclusions:** Over the last decades there has been a steadily increasing interest in the development and application of nonlinear time series models. In this study we focus on the widely used family of smooth transition autoregressive models, which appear to parsimoniously capture the nonlinear dependence of many economic and financial time series. Specifically, we illustrate the flexibility of the ESTAR model to encompass a number of different characteristics found in empirical work and suggested by the theoretical literature using market frictions and heterogeneous agents. These are multiple equilibria, complex dynamics, chaotic-like behavior, and spurious trends. In turn, we examine the properties of a recently proposed unit root test against smooth transition stationary processes when there are no priors regarding the deterministic components and possible multiple equilibria. We also make comparison with two alternative unit root tests widely employed in the literature. Our results stress that the power of the tests is highly dependent on the properties of the series. Moreover, no test dominates the others. Finally, we run several applications on the foreign exchange, stock and housing markets as well as a hyper-inflation series. Despite the fact that we can successfully fit nonlinear models implying a stable nonlinear adjustment mechanism which supports arbitrage conditions as well as globally stationary multiple equilibria inflation series, unit root tests fail, in general, to detect the mean reversion. Overall, our findings suggest that the difficulty to reject the unit root property in many financial and macroeconomic

data on the basis of unit root testing should not be regarded as conclusive evidence. In particular, the factors examined here can severely contaminate the power of both linear and nonlinear unit root tests.

**References:**

1. Baillie, Richard T., Ching-Fan Chung, and Margie A. Tieslau, "Analysing Inflation by the Fractionally Integrated ARFIMA-GARCH Model," *Journal of Applied Econometrics*, 1996, 11 (1), 23–40.
2. Benhabib, Jess, Stephanie Schmitt-Grohe, and Martin Uribe, "Monetary Policy and Multiple Equilibria," *American Economic Review*, 2001, 91 (1), 167–186.
3. Berka, Martin, "General Equilibrium Model of Arbitrage Trade and Real Exchange Rate Persistence," MPRA Paper 234, University Library of Munich, Germany 2005.
4. Black, Angela, Patricia Fraser, and Martin Hoesli, "House Prices, Fundamentals and Bubbles," *Journal of Business Finance & Accounting*, 2006, 33 (9-10), 1535–1555.
5. Brock, W.A., "Nonlinearity And Complex Dynamics In Economics And Finance," Working papers, Wisconsin Madison - Social Systems 1988. Byers, David J. and David A. Peel, "Non-linear Dynamics of Inflation in High Inflation Economies," *Manchester School*, 2000, 68 (0), 23–37.
6. Campbell, John Y. and Robert J. Shiller, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1988, 1 (3), 195–228.
7. Chapell, D., "Chaotic behaviour in a simple model of inflation," *Chaotic behaviour in a simple model of inflation*, 1997, LXV, 235–243.